Cavity photon counting: ab-initio derivation of the quantum jump superoperators and comparison of the existing models

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Abstract

Time development of electromagnetic fields in closed cavities under continuous detection of photons continues to be a subject of confusing controversy. Recently Dodonov *et al.* [Phys. Rev. A, **75**, 013806, 2007] argued that their model of quantum superoperators (E model) invalidates some of the predictions of the previously introduced photon counting model of Srinivas and Davies [J. Mod. Optic. **28**, 981, 1981] (SD model). Both the SD and the E models are based on two postulated quantum jump superoperators: (1) the one-count operator corresponding to the absorption of a single photon and (2) the no-count operator. In this work we develop a stochastic difference equation that describes the dissipative coupling of the cavity field and the detector. The difference equation is based on non-perturbative treatment of the cavity-detector coupling. In spite of being non-integrable due to the coupling of the detector with an external reservoir it can be used to derive the exact forms of the quantum jump superoperators. When applied to a particular photon counting measurement our theory gives predictions identical with those of the SD model which should be considered a *non-perturbative* and *ab-initio* result. It is pointed out that available experimental results coincide with the results given by the ab-initio SD model. We summarize some of the key characteristics of cavity fields and photon counting processes to demonstrate that the results given by the SD model are consistent with the principles of quantum mechanics while those given by the E model are not.

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I. INTRODUCTION

The theory of cavity photon counting and in particular the expected evolution of the cavity field after detection of a sequence of photons continues to be a subject of confusing controversy as shown in a recent report by Dodonov *et al.* [1]. So far no experimental data on the evolution of the cavity fields are available to resolve the controversy . The experiments themselves should be feasible at present though. This has prompted us to study the relation between the photon counting models. In this work we consider a system formed by the cavity field, the detector, and a (infinite) reservoir that is coupled to the detector.

The first quantum mechanical models of photon counting were formulated for an experiment where a light beam entered the detector and the unabsorbed photons were able to escape [2, 3]. Those models were based on the quantum version of the classical Mandel's formula [4]. In these models there was thus no need to consider the measurement back action. However, in the case of a cavity field the time integrated effect of the measurement on the electromagnetic field will be large and therefore the measurement back action cannot be ignored. We will show that the analysis of photon detection measurements can be carried out starting from the interaction Hamiltonian of the field and the detector. It is pointed out that for the photon counting analysis there is no need of introducing new postulates beyond the well established theoretical foundations of cavity quantum electrodynamics. It turns out that our non-perturbative approach gives results which agree with the SD model [4] while the E model [1, 5, 6, 7] is shown to be inconsistent. Furthermore, we show that the measured data of the second order coherence degrees of thermal and coherent fields [8] agree with the SD model.

II. PHOTON COUNTING BASED ON OPEN SYSTEM MASTER EQUATION

The SD and E photon counting models were originally introduced using phenomenological arguments or postulates. The first rigorous approach to cavity photon counting was formulated by Imato *et al.* [9]. They used a homogeneous atomic beam as a tool for including the dissipative terms in the Jaynes-Cummings Hamiltonian. The dissipation of the field was modeled using a sequence of infinitesimal perturbations caused by the atoms.

In this work we show that the atomic beam model is not needed for derivation of the

theory of cavity photon counting. Instead we assume that the whole system consists of three parts: the cavity, the detector and a (infinite) reservoir. The details of the coupling between detector and reservoir are unknown but it is assumed that this coupling is strong enough so that the detector is most of the time in its ground state $|g\rangle$. The time the detector will spend in its lowest excited state $|e\rangle$ is so short that the probability of a cavity photon to interact with the detector in the excited state is vanishingly small. Accordingly in any short time interval only absorption of one photon or no photons are possible i.e. the one-count or no-count events, respectively. The photoabsorption processes are Markovian processes i.e. the system is memoryless and the absorption probability depends only on the current state of the system, not on the past states. Furthermore, we assume that the detector doesn't return to the ground state by emitting the photon back to the cavity.

The optical cavity is assumed ideal and thus the only dissipative mechanism included is the absorption of photons by the detector. Furthermore, because the detector is assumed to return back to the ground state immediately after the absorption of a photon we do not consider the detector saturation effects. If we assume that the photons escape from the cavity to a detector which is, for example, a photomultiplier tube, the photons would dissipate from the cavity to the photomultiplier tube even if the tube was be momentarily switched off. Thus we can neglect the dead time of the detector.

We assume that the interaction of the field and the detector is described by the Jaynes-Cummings Hamiltonian in the rotating wave approximation

$$\hat{H}_I = \hbar\Omega\Big(\hat{a}^\dagger|g\rangle\langle e| + \hat{a}|e\rangle\langle g|\Big). \tag{1}$$

Furthermore, the field Hamiltonian is $\hbar\omega\hat{a}^\dagger\hat{a}$, the detector Hamiltonian is $\hbar\omega|e\rangle\langle e|$, and the initial detector-field density operator is

$$\hat{\rho}_I(0) = \sum_{n,n'=0}^{\infty} p_{n,n'}|n,g\rangle\langle n',g|. \tag{2}$$

We next find an expression for the differential change of the detector-field density operator $\mathrm{d}\hat{\rho}_I = \hat{\rho}_I(\mathrm{d}t) - \hat{\rho}_I(0)$. At the starting point of the interval $[0, \mathrm{d}t]$ the detector is in the ground state and the probability of the detector being excited to $|e\rangle$ is small. Therefore during $[0, \mathrm{d}t]$ the coupling of the detector with the reservoir can be neglected since the reservoir only causes relaxation of the detector from the excited state to the ground state. Thus the field detector subsystem can be considered closed within $[0, \mathrm{d}t]$. The differential change of the density

operator can then be obtained *non-perturbatively* using the Taylor series

$$\hat{\rho}_I(dt) - \hat{\rho}_I(0) \;=\; \left.\frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t}\right|_{t=0} \mathrm{d}t + \frac{1}{2}\left.\frac{\mathrm{d}^2\hat{\rho}_I(t)}{\mathrm{d}t^2}\right|_{t=0} (\mathrm{d}t)^2 + \dots . \tag{3}$$

As shown in detail in the Appendix A 1, the Taylor expansion gives (see Appendix A 1.)

$$\frac{\hat{\rho}_I(\mathrm{d}t) - \hat{\rho}_I(0)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\hat{H}_I, \hat{\rho}_I(0)] + \frac{\mathrm{d}t}{2\hbar^2}\left(2\hat{H}_I\hat{\rho}_I(0)\hat{H}_I - \{\hat{H}_I\hat{H}_I, \hat{\rho}_I(0)\}\right). \tag{4}$$

Note that this result is exact in the limit of infinitesimal dt. From equations (1) and (2) we see that $\hat{H}_I\hat{\rho}_I(0) = \hbar\Omega\sum_{n,n'} p_{n,n'}\sqrt{n}|n-1,e\rangle\langle n',g| = \hbar\Omega\hat{a}|e\rangle\langle g|\hat{\rho}_I(0)$, that $\hat{H}_I\hat{H}_I\hat{\rho}_I(0) = (\hbar\Omega)^2\sum_{n,n'} p_{n,n'}n|n,g\rangle\langle n',g| = (\hbar\Omega)^2\hat{a}^\dagger\hat{a}\hat{\rho}_I(0)$, and that $\hat{H}_I\hat{\rho}_I(0)\hat{H}_I = (\hbar\Omega)^2\sum_{n,n'} p_{n,n'}\sqrt{nn'}|n-1,e\rangle\langle n'-1,e| = (\hbar\Omega)^2\hat{a}|e\rangle\langle g|\hat{\rho}_I(0)|g\rangle\langle e|\hat{a}^\dagger$. Using these relations and replacing the initial time $t=0$ with t, equation (4) can be written as

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t} &= \frac{\hat{\rho}_I(t+\mathrm{d}t) - \hat{\rho}_I(t)}{\mathrm{d}t} \\ &= -\mathrm{i}\Omega\sum_{n,n'} p_{n,n'}(t)\Big(\sqrt{n}|n-1,e\rangle\langle n',g| - \sqrt{n'}|n,g\rangle\langle n'-1,e|\Big) \\ &\quad + \frac{\Omega^2\mathrm{d}t}{2}\Big(2\hat{a}|e\rangle\langle g|\hat{\rho}_I(t)|g\rangle\langle e|\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \hat{\rho}_I(t)\}\Big).\end{aligned} \tag{5}$$

Equation (5) can not be solved by simple integration since it has been assumed that in the beginning of each time interval $[t, t+\mathrm{d}t]$ the detector is at ground state and therefore the coupling between the the detector and the reservoir has been neglected. With these assumptions equation (5) describes the field-detector system as an open quantum system dissipating its energy to an infinite reservoir represented by the detector returning to the ground state infinitely fast. In the following we are interested in the evolution of the field density operator. This is obtained from equation (5) by calculating the trace over the detector states $\hat{\rho}_f = \langle g|\hat{\rho}_I|g\rangle + \langle e|\hat{\rho}_I|e\rangle$. Setting $\mathrm{d}\hat{\rho}_f = \hat{\rho}_f(t+\mathrm{d}t) - \hat{\rho}_f(t)$ and assuming that the detector is at ground state at t, the reduced density operator i.e. the density operator of the field can be written as

$$\frac{\mathrm{d}\hat{\rho}_f(t)}{\mathrm{d}t} = \frac{\Omega^2\mathrm{d}t}{2}\Big(2\hat{a}\hat{\rho}_f(t)\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \hat{\rho}_f(t)\}\Big), \tag{6}$$

Note that in contrast to equation (5), equation (6) may be solved by integration for our detector model, since it no longer includes the detector states. Substituting $\hat{\rho}_f(t) =$

$\sum_{n,n'=0}^{\infty} p_{n,n'}(t)|n\rangle\langle n'|$ into equation (6) gives

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_f(t)}{\mathrm{d}t} &= \sum_{n,n'=0}^{\infty} \frac{\mathrm{d}p_{n,n'}(t)}{\mathrm{d}t}|n\rangle\langle n'| \\ &= \Omega^2 \mathrm{d}t\Big(\sum_{n,n'=0}^{\infty} \sqrt{nn'}p_{n,n'}(t)|n-1\rangle\langle n'-1| - \sum_{n,n'=0}^{\infty} \frac{n+n'}{2}p_{n,n'}(t)|n\rangle\langle n'|\Big). \quad (7)\end{aligned}$$

In order to relate the constant $\Omega^2\mathrm{d}t$ to the coupling strength of the field-detector we solve the differential equation for the probabilities $p_{n,n}$ i.e. the master equation of the photon numbers. Taking the diagonal matrix elements $\langle n|\cdot|n\rangle$ of equation (7) and denoting $p_n = p_{n,n}$ gives

$$\begin{aligned}\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} &= \frac{\mathrm{d}\langle n|\hat{\rho}_f(t)|n\rangle}{\mathrm{d}t} \\ &= \Omega^2\mathrm{d}t\big((n+1)p_{n+1} - np_n\big). \quad (8)\end{aligned}$$

Thus, the detector only absorbs photons and do not emit them to the cavity. The constant $\lambda \equiv \Omega^2\mathrm{d}t$ is the probability per unit time that a photon will be absorbed by the detector i.e. the rate of absorption.

The physical interpretation of $\Omega^2\mathrm{d}t$ becomes particularly transparent if one considers the absorption of a single photon by the detector. Let the amplitude of the detector ground state + one photon field state ($|g,1\rangle$) be given by $C_g(t)$ and the amplitude of the excited state ($|e,0\rangle$) by $C_e(t)$ with initial conditions $C_g(0) = 1$ and $C_e(0) = 0$, respectively. The differential change of $C_e(t)$ can be then obtained from the time dependent Schrödinger equation. It is given by $C_e(\mathrm{d}t) = -\mathrm{i}V_{e,g}\mathrm{d}t/\hbar$, where $V_{e,g} = \langle e,0|\hat{H}_I|g,1\rangle = \hbar\Omega$ is the dipole amplitude. The probability that the photon has been absorbed is $|C_e(\mathrm{d}t)|^2 = |V_{e,g}|^2\mathrm{d}t^2/\hbar^2$. Thus the absorption probability per unit time is given by $|C_e(\mathrm{d}t)|^2/\mathrm{d}t = |V_{e,g}|^2\mathrm{d}t/\hbar^2 = \Omega^2\mathrm{d}t$. This is in accordance with equation (8).

III. ONE-COUNT AND NO-COUNT SUPEROPERATORS DERIVED FROM MASTER EQUATION

The density operator of the system after infinitesimal time $\mathrm{d}t$ is obtained using equation (5) and relation $\hat{\rho}_I(t+\mathrm{d}t) = \hat{\rho}_I(t) + \frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t}\mathrm{d}t$ which gives

$$\begin{aligned}\hat{\rho}_I(t+\mathrm{d}t) &= \hat{\rho}_I(t) + \Big[-\mathrm{i}\Omega\sum_{n,n'} p_{n,n'}(t)\Big(\sqrt{n}|n-1,e\rangle\langle n',g| - \sqrt{n'}|n,g\rangle\langle n'-1,e|\Big) \\ &\quad + \frac{\Omega^2\mathrm{d}t}{2}\Big(2\hat{a}|e\rangle\langle g|\hat{\rho}_I(t)|g\rangle\langle e|\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a},\hat{\rho}_I(t)\}\Big)\Big]\mathrm{d}t. \quad (9)\end{aligned}$$

After measuring a photoabsorption process the detector is in the excited state $|e\rangle$ so the part of the field density operator corresponding to the one-count event is given by $\langle e|\hat{\rho}_I(t+\mathrm{d}t)|e\rangle$. If a photon is not absorbed the detector stays in the ground state and the part of the field density operator corresponding to the no-count-event is given by $\langle g|\hat{\rho}_I(t+\mathrm{d}t)|g\rangle$. We obtain

$$\langle e|\hat{\rho}_I(t+\mathrm{d}t)|e\rangle = \lambda\hat{a}\hat{\rho}_f(t)\hat{a}^\dagger\mathrm{d}t \tag{10}$$

$$\langle g|\hat{\rho}_I(t+\mathrm{d}t)|g\rangle = \hat{\rho}_f(t) - \frac{\lambda}{2}\left(\hat{a}^\dagger\hat{a}\hat{\rho}_f(t) + \hat{\rho}_f(t)\hat{a}^\dagger\hat{a}\right)\mathrm{d}t. \tag{11}$$

Therefore, after the sudden absorption of a single photon the one-count operator is simply given by equation (10)

$$\hat{J}\hat{\rho}_f(t) = \lambda\hat{a}\hat{\rho}_f(t)\hat{a}^\dagger. \tag{12}$$

In contrast to the abrupt one-count process the no-count process changes the field density operator in a continuous manner. Therefore it is necessary to calculate the no-count operator in a finite time interval $[t, t+\tau]$ between two absorption events. The density operator is changed by the no-count event by the amount $\mathrm{d}\hat{\rho}_f(t) = -\frac{\lambda}{2}\{\hat{a}^\dagger\hat{a}, \hat{\rho}_f(t)\}\mathrm{d}t$, obtained from equation (11), whose solution in time interval $[t, t+\tau]$ (here τ is not necessarily infinitesimal) gives the no-count operator

$$\hat{S}_\tau\hat{\rho}_f(t) = e^{-\frac{\lambda}{2}\hat{a}^\dagger\hat{a}\tau}\hat{\rho}_f(t)e^{-\frac{\lambda}{2}\hat{a}^\dagger\hat{a}\tau}. \tag{13}$$

The probabilities of absorbing a photon and not absorbing a photon are obtained from equations (12) and (13), respectively, by calculating $\mathrm{Trace}\{\hat{J}\hat{\rho}_I\}/\mathrm{Trace}\{\hat{\rho}_I\}$ and $\mathrm{Trace}\{\hat{S}_\tau\hat{\rho}_I\}/\mathrm{Trace}\{\hat{\rho}_I\}$. Equation (12) gives the probability of the abrupt absorption process at $[t, t+\mathrm{d}t]$

$$P_{one-count}(t, t+\mathrm{d}t) = \lambda\bar{n}(t)\mathrm{d}t. \tag{14}$$

The no-count probability at time interval $[t, t+\tau]$ is obtained calculating the trace of (13). Using the series expansion of the exponential terms gives (see Appendix A 4)

$$P_{no-count}(t, t+\tau) = \sum_{n=0}^{\infty} e^{-\lambda n\tau}p_n(t). \tag{15}$$

Note that for infinitesimal time interval $\tau = \mathrm{d}t$ one obtains $\sum_{n=0}^{\infty} e^{-\lambda n\mathrm{d}t}p_n(t) = \sum_{n=0}^{\infty}(1 - \lambda n\mathrm{d}t)p_n(t) = 1 - \lambda\bar{n}(t)\mathrm{d}t$ i.e. the sum of the one-count and no-count probabilities is unity.

The change in the expectation value of the number of photons in a small time interval $\mathrm{d}t$ is obtained from equation (8)

$$\begin{aligned}\frac{\mathrm{d}\bar{n}(t)}{\mathrm{d}t} &= \sum_{n=0}^{\infty} n\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \lambda\sum_{n=0}^{\infty} n\left(-np_n(t)+(n+1)p_{n+1}(t)\right) \\ &= -\lambda\sum_{n=0}^{\infty} np_n(t) = -\lambda\bar{n}(t) \end{aligned} \tag{16}$$

which gives

$$\bar{n}(t) = \bar{n}(0)e^{-\lambda t}. \tag{17}$$

Thus the expectation value of the number of photons decays exponentially in time.

Note the difference between our model and the model of Imato *et al* [9]. By using the perturbation theory, they calculated how an atom perturbs the cavity field during an infinitesimal interaction time and thereby dissipates the photons from the cavity. In contrast we have derived, starting from the time-dependent Schrödinger equation, a master equation for a system where the cavity field is coupled with a detector which relaxate to a infinite reservoir. However, both models leads to the SD model when the detector states are reduced.

IV. COMPARISON OF PHOTON COUNTING MODELS

The SD and the E models share four main postulates:

(1) The absorption of photons the detector takes place as instantaneous events represented by the one-count superoperator $\hat{J}_A\hat{\rho}_f(t) = \gamma_A\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger$ (γ_A is defined in postulate 4). The one-count operator is a quantum jump superoperator i.e. the density operator jumps from $\hat{\rho}_f(t)$ to $\gamma_A\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger$ in infinitesimal time interval $[t, t+\mathrm{d}t]$. In the SD model $\hat{A}\equiv\hat{a}$. The E model is obtained from the SD model by replacing the well known bosonic annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators by the normalized operators $\hat{A}\equiv(\hat{a}^\dagger\hat{a}+1)^{-1/2}\hat{a}$ (denoted by $\hat{E}$ below) and $\hat{A}^\dagger\equiv\hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)^{-1/2}$ (denoted by $\hat{E}^\dagger$ below), respectively [5]. These normalized operators obey the relations $\hat{E}|0\rangle = 0$, $\hat{E}|n>0\rangle = |n-1\rangle$ and $\hat{E}^\dagger|n\rangle = |n+1\rangle$. The probability of the one-count at $[t, t+\mathrm{d}t]$ is $\gamma_A\mathrm{Trace}\{\hat{A}\hat{\rho}_f\hat{A}^\dagger\}\mathrm{d}t$.

(2) Between the counts the density operator evolves according to the no-count superoperator $\hat{S}_\tau\hat{\rho}_f = e^{\hat{Y}_A\tau/\hbar}\hat{\rho}_f(t)e^{\hat{Y}_A{}^\dagger\tau/\hbar}$, where $\hat{Y}_A = -\mathrm{i}\hat{H}_0 - \frac{1}{2}\hbar\gamma_A\hat{A}^\dagger\hat{A}$ and $\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a}$. Here τ is not necessarily differential.

(3) After measuring an event corresponding to the operator $\hat{\mathcal{O}}$ ($\hat{\mathcal{O}} = \hat{J}$ or $\hat{S}$), the density

operator is $\hat{\rho}_f(t^+) = \hat{\mathcal{O}}\hat{\rho}_f(t)/\text{Trace}\{\hat{\mathcal{O}}\hat{\rho}_f(t)\}$.

(4) Furthermore, the coupling between the detector and the field is parameterized using a model dependent coupling coefficient γ. The γ_{sd} and γ_e are not necessary equal but if we equate the absorption rates from one photon Fock states (i.e. $|1\rangle$) we obtain that $\gamma_{sd} = \gamma_e$.

The one-count operator (Eq. (12)) and the probability (Eq. (14)) and the no-count operator (Eq. (13)) and the probability (Eq. (15)) given by our model are equivalent with those given by the SD model (see App. A 2 and A 4). Thus our model gives an ab-initio derivation for the initially postulated SD model.

Table I summarizes the expectation values of the number of photons after the one-count event, reported previously in references [6, 10]. For detailed derivations see Appendix A 3. Also the probabilities of observing the vacuum state after the one-count event are given. Note the difference between the models for the thermal and coherent fields. Our model agrees for all calculated expectation values with results given by the SD-form of superoperators while they disagree with the results given by the E model as seen in Table I.

TABLE I: The expectation values of the number of photons after a one-count event are given in the upper part. In the lower part the probabilities of the vacuum state after one-count event are given for the thermal and the coherent fields (p_0 and p_1 are the probabilities of the zero and one photon number states, respectively). If the field is initially in the Fock state $|1\rangle$ the probability $p_0(t^+) = 1$ after the one-count event, otherwise $p_0(t^+) = 0$.

Initial state	SD model	E model
Fock	$\bar{n}(t^+)=\bar{n}(t)-1$	$\bar{n}(t^+)=\bar{n}(t)-1$
Thermal	$\bar{n}(t^+)=2\bar{n}(t)$	$\bar{n}(t^+)=\bar{n}(t)$
Coherent	$\bar{n}(t^+)=\bar{n}(t)$	$\bar{n}(t^+)=\frac{\bar{n}(t)}{1-e^{-\bar{n}(t)}}-1$
Thermal	$p_0(t^+)=\frac{1}{(1+\bar{n}(t))^2}$	$p_0(t^+)=\frac{1}{1+\bar{n}(t)}$
Coherent	$p_0(t^+)=e^{-\bar{n}(t)}$	$p_0(t^+)=\frac{\bar{n}(t)}{e^{\bar{n}(t)}-1}$

Table II gives photon count rates for the SD and the E models. The results in Table II give the probability $w^{(1)}(t)\mathrm{d}t$ of counting one photon at $[t, t+\mathrm{d}t]$ and the conditional probability $w^{(1)}(t^+|t)\mathrm{d}t$ of counting one photon at $[t, t+\mathrm{d}t]$ and the second photon at $[t^+, t^+ + \mathrm{d}t]$ ($t^+ = t + \mathrm{d}t$ is infinitesimally greater than t and $\mathrm{d}t$ is the infinitesimal time that the one-

count process takes i.e. we are calculating the probability of absorbing the second photon immediately after the first photon). Furthermore, the second order coherence degree

$$g^{(2)}(t,t^+) = \frac{w^{(1)}(t^+|t)}{w^{(1)}(t^+)} \tag{18}$$

is also given in Table II. For the derivation of the photon correlation function see Appendix A 7. Our results agree with the results of the SD model.

TABLE II: The one-count rates ($w^{(1)}(t)$), conditional one-count rates ($w^{(1)}(t^+|t)$), and the second order coherence degrees ($g^{(2)}(t,t^+)$) given by the SD and the E models.

Initial state	$w_{sd}^{(1)}(t)$	$w_{sd}^{(1)}(t^+\|t)$	$w_e^{(1)}(t)$	$w_e^{(1)}(t^+\|t)$	$g_{sd}^{(2)}(t,t^+)$	$g_e^{(2)}(t,t^+)$
Fock	$\gamma_{sd}\bar{n}(t)$	$\gamma_{sd}(\bar{n}(t)-1)$	γ_e	γ_e	$\frac{\bar{n}(t)-1}{\bar{n}(t)}$	1
Thermal	$\gamma_{sd}\bar{n}(t)$	$2\gamma_{sd}\bar{n}(t)$	$\gamma_e\frac{\bar{n}(t)}{1+\bar{n}(t)}$	$\gamma_e\frac{\bar{n}(t)}{1+\bar{n}(t)}$	2	1
Coherent	$\gamma_{sd}\bar{n}(t)$	$\gamma_{sd}\bar{n}(t)$	$\gamma_e(1-e^{-\bar{n}(t)})$	$\gamma_e\left(1-\frac{\bar{n}(t)}{e^{\bar{n}(t)}-1}\right)$	1	$\frac{e^{\bar{n}(t)}-(\bar{n}(t)+1)}{e^{\bar{n}(t)}+e^{-\bar{n}(t)}-2}$

The evolution of the expectation value of the number of photons in the SD and E models are respectively (see Appendix A 6)

$$\bar{n}_{sd}(t) = \bar{n}(0)e^{-\gamma_{sd}t} \tag{19}$$

$$\bar{n}_e(t) = \bar{n}(0) + \gamma_e \int_0^t (p_0(t') - 1)\mathrm{d}t'. \tag{20}$$

Thus $\bar{n}(t)$ decays exponentially in the SD model irrespective of the field type while in the E model $\bar{n}(t)$ depends on the time integral of the vacuum state probability which is different for different field types. Again our results agree with the results of the SD model.

V. DISCUSSION

The E model of photon counting was originally motivated by arguing that the results given by the SD model are unphysical. Next we will show by selected examples that, on the contrary, it is the SD model that gives physical and also intuitionally right answers.

In references [1, 6, 7] Dodonov *et al.* gave the following main arguments in favor of the E model: (1) The expectation value of the number of photons may increase when operating with the SD one-count operator [10], see Table I. (2) The absorption rate of photons in the SD model is proportional to the number of photons and does not saturate for high $\bar{n}$.

1. *Increase of $\bar{n}$ after one-count event*

In the SD model the expectation value of the number of photons after the observed absorption of one photon (i.e. after the one-count event) may be greater than before the absorption. For example after detecting of a photon with the SD one-count operator from the thermal field the expectation value of the number of photons doubles (see Table I). The assumption that this feature of the SD model is unphysical was the main justification for the E model. This behavior caused by the measurement back action to the cavity field is, however, in agreement with the photon bunching effect: When a photon arrives to a detector the expectation value of the number of photons raises and thus it is more probable to detect another photon. The growth of the expectation value of the number of photons is not unphysical and means that the states which had one or more photons become more probable after the absorption of the first photon.

Furthermore, we have shown that in our model (as well as in the SD model) the expectation value of the number of the photons has exponential decay in time for all fields as it should. Thus, even if the expectation value of the number of photons may increase in the one-count event, on average the expectation value of the number of the photons decreases on every time interval during which no measurement induced projection of the field state occurs.

2. *Saturation of absorption rate*

The one-count rates for the SD and E models are $w_{sd}^{(1)}(t) = \gamma_{sd}\bar{n}(t)$ and $w_{e}^{(1)}(t) = \gamma_e(1 - p_0(t))$, respectively. The fundamental difference between the models is that in the SD model the photocount rate is proportional to the expectation value of the number of photons while in the E model it is proportional to the probability that photons exists. Thus the absorption probability saturates in the E model. This may be a reasonable assumption for the very high intensities when the detector itself may saturate. However, the possible saturation of absorption probability is a internal property of the detector and should be an auxiliary model property, not a built in property of the quantum jump superoperators. In principle it is possible that at high intensities the detector could scatter some photons back to the cavity. This would lead to non-constant γ which the present models (SD or E) can not adopt

as such.

Let us consider two simple examples: (a) The field is in the state $|\Psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The one-count rates are $w_{sd}^{(1)}(t) = \gamma_{sd}/2$ and $w_e^{(1)}(t) = \gamma_e/2$. (b) The field is the state $|\Psi\rangle = (|0\rangle + |100\rangle)/\sqrt{2}$. The one-count rates are $w_{sd}^{(1)}(t) = 100\gamma_{sd}/2$ and $w_e^{(1)}(t) = \gamma_e/2$. The behavior of the SD model seems more reasonable because it should be more probable to detect photons when the expectation value of the photons is larger.

3. Second order coherence degree

The photon bunching effect is related to the second order coherence: If $0 \leq g^{(2)}(0) < 1$ the light is antibunched, if $g^{(2)}(0) = 1$ the light is nonbunched or random, and if $g^{(2)}(0) > 1$ the light is bunched [11]. Here the argument zero means that the correlation is measured with zero time delay. Examples of light fields obeying the above conditions are the Fock state, the coherent state and the thermal state, respectively. The calculated second order coherence degrees are given in Table II.

For single-mode fields the second order coherence degrees are independent of the time delay between the measurements. The values are $g^{(2)} = 1$ for the coherent field, $g^{(2)} = 2$ for the thermal field and $g^{(2)} = m/(m-1)$ for the Fock state $|m\rangle$ $(m \geq 2)$ [11]. Comparing the results given in Table II we note that the SD model reproduces the second order coherence degrees of the three standard example fields.

The results given by our model (as well as the SD model) are equal with the experimental results given in Reference [8] for thermal and coherent fields. Note, however, that the measurement results in Reference [8] are not given for light fields in a closed cavity but for continuous wave laser with a Gaussian frequency distribution for thermal light. We are comparing the results measured with zero time delay since the zero time delay cancels the multi-mode effect. Furthermore, we expect that the measurement results of the second order coherence degree in free space correspond to our theoretical results since $g^{(2)}$ depends only on the statistics of the light field, not the spatial distribution.

Dodonov *et al.* [6] also concluded that the SD model always predicts the photon bunching phenomenon for any initial field. We have shown (see Table II) that this conclusion given in reference [6] is incorrect.

VI. CONCLUSIONS

In this paper we have shown that by starting from the time-dependent Schrödinger equation one can derive an exact difference equation for the density operator $\hat{\rho}_I$ of the combined detector-field system. Reducing this change of the density operator with respect to the detector variables gives a time integrable master equation for the field density operator. This gives the first principle expressions (Eqs. (12) and (13)) for the one-count and no-count operators. Our results for the one-count and no-count operators agree with the SD model (see App. A 2 and A 4 for comparison) which therefore, in spite of being initially introduced ad hoc [4], is actually an exact quantum mechanical result.

APPENDIX A: DERIVATIONS

1. Derivation of master equation

The time-dependent Schrödinger equation is

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_s(t)\rangle = (\hat{H}_0^s + \hat{H}_I^s)|\Psi_s(t)\rangle, \tag{A1}$$

where $\hat{H}_0^s = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar\omega_A|e\rangle\langle e|$ is the time-independent field + detector Hamiltonian and $\hat{H}_I^s = \hbar\Omega\left(\hat{a}|e\rangle\langle g| + \hat{a}^\dagger|g\rangle\langle e|\right)$ is the time-independent interaction Hamiltonian. The interaction representation is given by unitary transformation

$$|\Psi_I(t)\rangle = e^{\mathrm{i}\hat{H}_0^s t/\hbar}|\Psi_s(t)\rangle, \tag{A2}$$

where $|\Psi_I(0)\rangle = |\Psi_s(0)\rangle$. The Schrödinger equation gives

$$\begin{aligned}\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_I(t)\rangle &= \mathrm{i}\hbar\left(\frac{\mathrm{i}\hat{H}_0^s}{\hbar}|\Psi_I(t)\rangle + e^{\mathrm{i}\hat{H}_0^s t/\hbar}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_s(t)\rangle\right)\\ &= -\hat{H}_0^s|\Psi_I(t)\rangle + e^{\mathrm{i}\hat{H}_0^s t/\hbar}(\hat{H}_0^s + \hat{H}_I^s)e^{-\mathrm{i}\hat{H}_0^s t/\hbar}|\Psi_I(t)\rangle\\ &= e^{\mathrm{i}\hat{H}_0^s t/\hbar}\hat{H}_I^s e^{-\mathrm{i}\hat{H}_0^s t/\hbar}|\Psi_I(t)\rangle\\ &= \hat{H}_I^s|\Psi_I(t)\rangle,\end{aligned} \tag{A3}$$

where the last form holds, since $[\hat{H}_0^s, \hat{H}_I^s] = 0$. Note that, since $\hat{H}_0^s$ and $\hat{H}_I^s$ commute, the interaction Hamiltonian in the interaction picture is same as in the Schrödinger picture.

Equation (A3) gives following differential equation for the density operator of the coupled field-detector system in the interaction picture

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t} = [\hat{H}_I^s, \hat{\rho}_I(t)]. \tag{A4}$$

The density operator can be written using the Taylor series as

$$\hat{\rho}_I(dt) \;=\; \hat{\rho}_I(0) + \left.\frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t}\right|_{t=0} \mathrm{d}t + \frac{1}{2}\left.\frac{\mathrm{d}^2\hat{\rho}_I(t)}{\mathrm{d}t^2}\right|_{t=0} (\mathrm{d}t)^2 + \ldots. \tag{A5}$$

Since the interaction Hamiltonian is time independent, equation (A4) can be used to obtain the higher derivatives in the Taylor series

$$\frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t} \;=\; -\frac{\mathrm{i}}{\hbar}[\hat{H}_I^s, \hat{\rho}_I(t)] \tag{A6}$$

$$\begin{aligned}\frac{\mathrm{d}^2\hat{\rho}_I(t)}{\mathrm{d}t^2} &= -\frac{\mathrm{i}}{\hbar}[\hat{H}_I^s, \frac{\mathrm{d}\hat{\rho}_I(t)}{\mathrm{d}t}] \\ &= -\frac{1}{\hbar^2}\left[\hat{H}_I^s, [\hat{H}_I^s, \hat{\rho}_I(t)]\right].\end{aligned} \tag{A7}$$

Taking the terms up to second order in $\mathrm{d}t$ the Taylor expansion can be written as

$$\hat{\rho}(dt) \;=\; \hat{\rho}_I(0) - \frac{\mathrm{i}}{\hbar}[\hat{H}_I^s, \hat{\rho}_I(0)]\mathrm{d}t - \frac{1}{2\hbar^2}\left[\hat{H}_I^s, [\hat{H}_I^s, \hat{\rho}_I(0)]\right](\mathrm{d}t)^2, \tag{A8}$$

where the double commutator is $\left[\hat{H}_I^s, [\hat{H}_I^s, \hat{\rho}_I(0)]\right] = [\hat{H}_I^s, \hat{H}_I^s\hat{\rho}_I(0) - \hat{\rho}_I(0)\hat{H}_I^s] = \hat{H}_I^s\hat{H}_I^s\hat{\rho}_I(0) + \hat{\rho}_I(0)\hat{H}_I^s\hat{H}_I^s - 2\hat{H}_I^s\hat{\rho}(0)\hat{H}_I^s$ giving

$$\hat{\rho}(dt) \;=\; \hat{\rho}_I(0) - \frac{\mathrm{i}}{\hbar}[\hat{H}_I^s, \hat{\rho}_I(0)]\mathrm{d}t + \frac{1}{2\hbar^2}\left(2\hat{H}_I^s\hat{\rho}(0)\hat{H}_I^s - \{\hat{H}_I^s\hat{H}_I^s, \hat{\rho}_I(0)\}\right). \tag{A9}$$

2. One-count probabilities and rates

The one-count probability in the time interval $[t, t+\mathrm{d}t]$ is given by $\gamma_A \mathrm{Trace}\{\hat{A}^\dagger\hat{A}\hat{\rho}_f\}\mathrm{d}t$. Thus $\gamma_A \mathrm{Trace}\{\hat{A}^\dagger\hat{A}\hat{\rho}_f\}$ is the photon count rate. For the SD and E models we, respectively, obtain the photon count rates ($\mathrm{Trace}\{\cdot\} = \sum_{n=0}^{\infty}\langle n|\cdot|n\rangle$)

$$w_{sd}(t) \;=\; \gamma_{sd}\sum_{n=0}^{\infty} np_n(t) = \gamma_{sd}\bar{n}(t) \tag{A10}$$

$$w_e(t) \;=\; \gamma_e\sum_{n=1}^{\infty} p_n(t) = \gamma_e(1 - p_0(t)) \tag{A11}$$

The one-count probabilities are obtained by multiplying the rates with $\mathrm{d}t$. Thus $P_{sd}^{one-count}(t) = \gamma_{sd}\bar{n}(t)\mathrm{d}t$ and $P_e^{one-count}(t) = \gamma_e(1 - p_0(t))\mathrm{d}t$.

3. One-count event

If the one-count event is detected the density operator must change in accordance to the operation by the one-count operator and normalization. The density operator after one-count is $\hat{\rho}_f(t^+) = \frac{\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger}{\text{Trace}\{\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger\}} = \frac{\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger}{\text{Trace}\{\hat{A}^\dagger\hat{A}\hat{\rho}_f(t)\}}$. For the SD and E models we, respectively, obtain

$$\begin{aligned}\hat{\rho}_f^{sd}(t^+) &= \frac{1}{\bar{n}(t)}\sum_{n,n'=0}^{\infty} p_{n,n'}(t)\sqrt{nn'}|n-1\rangle\langle n'-1| \\ &= \frac{1}{\bar{n}(t)}\sum_{n=0}^{\infty} p_{n+1,n'+1}(t)\sqrt{(n+1)(n'+1)}|n\rangle\langle n'| \qquad \text{(A12)}\end{aligned}$$

$$\hat{\rho}_f^{e}(t^+) = \frac{1}{1-p_0(t)}\sum_{n,n'=1}^{\infty} p_{n,n'}(t)|n-1\rangle\langle n'-1| = \frac{\sum_{n,n'=0}^{\infty} p_{n+1,n'+1}(t)|n\rangle\langle n'|}{1-p_0(t)}. \qquad \text{(A13)}$$

Thus the new probabilities of the field states of n photons are

$$p_n^{sd}(t^+) = \frac{n+1}{\bar{n}(t)}p_{n+1}(t) \qquad \text{(A14)}$$

$$p_n^{e}(t^+) = \frac{p_{n+1}(t)}{1-p_0(t)}. \qquad \text{(A15)}$$

4. No-count probability

The no-count operator is $\hat{S}_\tau\hat{\rho}_f = e^{\hat{Y_A}\tau/\hbar}\hat{\rho}_f e^{\hat{Y_A}^\dagger\tau/\hbar}$, where $\hat{Y}_A = -\mathrm{i}\hat{H}_0 - \frac{1}{2}\gamma_A\hbar\hat{A}^\dagger\hat{A}$ and $\hat{H}_0 = \hbar\omega_0\hat{a}^\dagger\hat{a}$. The series expansion in the SD model is

$$e^{\left(-\mathrm{i}\omega_0-\frac{1}{2}\gamma\right)\hat{a}^\dagger\hat{a}\tau}|m\rangle = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}\omega_0-\frac{1}{2}\gamma)^n(\hat{a}^\dagger\hat{a})^n\tau^n}{n!}|m\rangle = e^{\left(-\mathrm{i}\omega_0-\frac{1}{2}\gamma\right)m\tau}|m\rangle \qquad \text{(A16)}$$

Thus the time evolution of the density operator in the no-count event is

$$e^{\hat{Y}\tau}\hat{\rho}_f e^{\hat{Y}^\dagger\tau} = \sum_{m,m'=0}^{\infty} e^{-\mathrm{i}\omega_0(m-m')\tau-\gamma\frac{m+m'}{2}\tau}p_{m,m'}|m\rangle\langle m'|.$$

The series expansion in the E model is

$$e^{\hat{Y}\tau} = e^{-\mathrm{i}\omega_0\hat{a}^\dagger\hat{a}\tau-\frac{1}{2}\gamma\hat{E}^\dagger\hat{E}\tau} = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}\omega_0)^n(\hat{a}^\dagger\hat{a})^n\tau^n}{n!}\sum_{k=0}^{\infty}\frac{(-\frac{1}{2}\gamma)^k(\hat{E}^\dagger\hat{E})^k\tau^k}{k!} \qquad \text{(A17)}$$

giving

$$e^{\hat{Y}\tau}|m\rangle = e^{-\mathrm{i}\omega_0 m\tau}\begin{cases}\sum_{k=0}^{\infty}\frac{(-\frac{1}{2}\gamma)^k\tau^k}{k!}|m\rangle \\ |0\rangle\end{cases} = \begin{cases}e^{-\mathrm{i}\omega_0 m\tau-\frac{1}{2}\gamma\tau}|m\rangle, & m>0 \\ |0\rangle, & m=0\end{cases} \qquad \text{(A18)}$$

and the evolution of the density operator

$$
\begin{aligned}
e^{\hat{Y}\tau}\hat{\rho}_f e^{\hat{Y}^\dagger\tau} &= \sum_{m,m'=0}^{\infty} e^{-\mathrm{i}\omega_0\hat{a}^\dagger\hat{a}\tau-\frac{1}{2}\gamma\hat{E}^\dagger\hat{E}\tau} p_{m,m'}|m\rangle\langle m'|e^{+\mathrm{i}\omega_0\hat{a}^\dagger\hat{a}\tau-\frac{1}{2}\gamma\hat{E}^\dagger\hat{E}\tau} \\
&= p_{0,0}|0\rangle\langle 0| + \sum_{m=1}^{\infty}\left[p_{0,m}|0\rangle\langle m|e^{+\mathrm{i}\omega_0 m\tau-\frac{1}{2}\gamma\tau} + p_{m,0}|m\rangle\langle 0|e^{-\mathrm{i}\omega_0 m\tau-\frac{1}{2}\gamma\tau}\right] \\
&\quad + \sum_{m,m'=1}^{\infty} p_{m,m'}e^{-\mathrm{i}\omega_0(m-m')\tau-\gamma\tau}|m\rangle\langle m'|. \qquad \text{(A19)}
\end{aligned}
$$

We are interested only on the diagonal elements to be able to calculate the evolution of the probability of the n photon state by taking the matrix elements $\langle n|\cdot|n\rangle$. Thus the SD model gives

$$
\langle n|\hat{S}_\tau\hat{\rho}_f(0)|n\rangle = e^{-n\gamma_{sd}\tau}p_n(0) \qquad \text{(A20)}
$$

and the E model gives

$$
\langle n|\hat{S}_\tau\hat{\rho}_f(0)|n\rangle = \begin{cases} e^{-\gamma_e\tau}p_n(0), & n>0 \\ p_0(0), & n=0 \end{cases} \qquad \text{(A21)}
$$

The no-count probabilities at the time interval $[t, t+\tau]$ are the sums over the probabilities $\langle n|\hat{S}_\tau\hat{\rho}_f(t)|n\rangle$. The SD model and the E model, respectively, give

$$
\begin{aligned}
P_{sd}^{no-count}(t,t+\tau) &= \sum_{n=0}^{\infty} e^{-n\gamma_{sd}\tau}p_n(t) \qquad \text{(A22)} \\
P_{e}^{no-count}(t,t+\tau) &= p_0(t) + (1-p_0(t))e^{-\gamma_e\tau}. \qquad \text{(A23)}
\end{aligned}
$$

5. No-count event

If we know that the one-count event has not happened, the density operator must change in accordance to the operation by the no-count operator and normalization. Thus the probabilities of n photon states are obtained by normalizing equations (A20) and (A21) giving for the SD and E models, respectively,

$$
\begin{aligned}
p_n(t+\tau) &= \frac{e^{-n\gamma_{sd}\tau}p_n(t)}{\sum_{n=0}^{\infty} e^{-n\gamma_{sd}\tau}p_n(t)} \qquad \text{(A24)} \\
p_n(t+\tau) &= \begin{cases} \frac{e^{-\gamma_e\tau}p_n(t)}{p_0(t)+(1-p_0(t))e^{-\gamma_e\tau}}, & n>0 \\ \frac{p_0(t)}{p_0(t)+(1-p_0(t))e^{-\gamma_e\tau}}, & n=0. \end{cases} \qquad \text{(A25)}
\end{aligned}
$$

6. Evolution of expectation value of photon number

The density matrix evolves according to equation [1, 4, 6]

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}\omega\left(\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) + \left(\gamma_A\hat{A}\hat{\rho}\hat{A}^\dagger - \frac{\gamma_A}{2}(\hat{A}^\dagger\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\hat{A})\right). \tag{A26}$$

The probabilities of n photon states are given by the diagonal elements $\langle n|\cdot|n\rangle$. Thus we obtain for the SD model

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \gamma_{sd}\left((n+1)p_{n+1}(t) - p_n(t)n\right) \tag{A27}$$

and for the E model

$$\frac{\mathrm{d}p_{n\geq 1}(t)}{\mathrm{d}t} = \gamma_e\left(p_{n+1}(t) - p_n(t)\right) \tag{A28}$$

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = \gamma_e p_1(t). \tag{A29}$$

The expectation value of the number of photons is defined as

$$\bar{n}(t) = \sum_{n=0}^{\infty} n p_n(t). \tag{A30}$$

For the SD model the time derivation of $\bar{n}(t)$ is given by

$$\begin{aligned}
\frac{\mathrm{d}\bar{n}(t)}{\mathrm{d}t} &= \sum_{n=0}^{\infty} n\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \gamma_{sd}\sum_{n=0}^{\infty}\left[(n+1)np_{n+1}(t) - n^2p_n(t)\right] \\
&= \gamma_{sd}\left(\sum_{n=0}^{\infty}(n+1)np_{n+1}(t) - \sum_{n=0}^{\infty}n^2p_n(t)\right) = \gamma_{sd}\left(\sum_{n=1}^{\infty}n(n-1)p_n(t) - \sum_{n=1}^{\infty}n^2p_n(t)\right) \\
&= -\gamma_{sd}\sum_{n=0}^{\infty}np_n(t) = -\gamma_{sd}\bar{n}(t).
\end{aligned}$$

This has a solution

$$\bar{n}(t) = \bar{n}(0)e^{-\gamma_{sd}t}. \tag{A31}$$

For the E model we correspondingly obtain

$$\begin{aligned}
\frac{\mathrm{d}\bar{n}(t)}{\mathrm{d}t} &= \sum_{n=0}^{\infty} n\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \gamma_e\sum_{n=0}^{\infty}\left[np_{n+1}(t) - np_n(t)\right] \\
&= \gamma_e\left(\sum_{n=0}^{\infty}np_{n+1}(t) - \sum_{n=0}^{\infty}np_n(t)\right) = \gamma_e\left(\sum_{n=1}^{\infty}(n-1)p_n(t) - \sum_{n=1}^{\infty}np_n(t)\right) \\
&= -\gamma_e\sum_{n=1}^{\infty}p_n(t) = -\gamma_e(1-p_0(t)).
\end{aligned}$$

Time integration gives for the expectation value of the number of photons a solution

$$\bar{n}(t) = \bar{n}(0) + \gamma_e\int_0^t (p_0(t') - 1)\mathrm{d}t'. \tag{A32}$$

7. Second order coherence degree

The second order coherence degree is [11, 12]

$$g^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2,\mathbf{r}_2,t_2,\mathbf{r}_1,t_1)=\frac{G^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2,\mathbf{r}_2,t_2,\mathbf{r}_1,t_1)}{G^{(1)}(\mathbf{r}_1,t_1,\mathbf{r}_1,t_1)G^{(1)}(\mathbf{r}_2,t_2,\mathbf{r}_2,t_2)}, \tag{A33}$$

where

$$\begin{aligned}&G^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2,\mathbf{r}_2,t_2,\mathbf{r}_1,t_1)\\ =\;&\mathrm{Tr}\{\hat{\rho}_f\hat{\mathbf{E}}^{(-)}(\mathbf{r}_1,t_1)\hat{\mathbf{E}}^{(-)}(\mathbf{r}_2,t_2)\hat{\mathbf{E}}^{(+)}(\mathbf{r}_2,t_2)\ldots\hat{\mathbf{E}}^{(+)}(\mathbf{r}_1,t_1)\}\end{aligned}$$

with $\hat{\mathbf{E}}^{(-)}(\mathbf{r},t)$ and $\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)$ being the negative and positive frequency parts of the electric field operator. The two-fold delayed coincidence rate i.e. the counting rate per (unit time)^2 is given by [12]

$$w^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2,\mathbf{r}_2,t_2,\mathbf{r}_1,t_1)=s^2G^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2,\mathbf{r}_2,t_2,\mathbf{r}_1,t_1), \tag{A34}$$

where s is the sensitivity of the detector. We consider only the temporal correlation so we assume that all of the position vectors are equal and drop the spatial coordinate. We can now use well know formula of conditional probability: the probability that an event B occurs with the condition that A has happened is $p(B|A)=p(B\cap A)/p(A)$. Thus $p(B\cap A)=p(B|A)p(A)$ giving $w^{(2)}(t^+,t)(\mathrm{d}t)^2 = w^{(1)}(t^+|t)\mathrm{d}t\; w^{(1)}(t)\mathrm{d}t$, where we are considering correlation with infinitesimal time difference. Furthermore, we can write the second order coherence degree using the count rates

$$g^{(2)}(t,t^+)\;=\;\frac{w^{(2)}(t,t^+)}{w^{(1)}(t)\;w^{(1)}(t^+)}=\frac{w^{(1)}(t^+|t)w^{(1)}(t)}{w^{(1)}(t)w^{(1)}(t^+)}=\frac{w^{(1)}(t^+|t)}{w^{(1)}(t^+)}, \tag{A35}$$

where we, furthermore, assume that $w^{(1)}(t^+)=w^{(1)}(t)$ due to the differential time difference.

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